The definite Gentzen Integral and thoughts on the indefinite one [preprint for IJAPL]

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Abstract

Gerhard Gentzen is mostly known for his ground-breaking work in proof theory, especially for his Hauptsatz. But what many people do not know: in his last days in Soviet prison he constructed his interpretation of a logical integral, which is referred to as the Gentzen integral in the forthcoming text. Because his notes are only partially available due to various reasons, we cannot know for sure how exactly it was defined. But what we do know is: with the notion of this integral, the whole discipline of logical analysis would have evolved in a different way.

In this text, it is presented what we were able to reconstruct from his definite Gentzen integral, which takes a semantic approach and answers the milleniums-old question of "how true is a statement?" in a precise way. This is done by a recursive definition for a construction that is mostly measure-theoretical. The resulting concept will not only work in the context of classical first order logic, but also in more powerful constructions as the Hegelian one. This generality alone would have revolutionized the philosophy of the last three centuries!

There still is much work to be done: we have reason to believe that Gentzen's understanding of logic was even more ahead of its time. Of his notebook that he used in his last days, only several sentences have survived. Even from these phrases, one can testify a definite of schizophrenia. But various formulae are retained as well, and in these, something like the calculus of our nowadays well-known pseudo-probabilistic differential is used for obtaining explicit antiderivative formulae. This indefinite Gentzen integral is much harder to reconstruct and we have not succeeded yet do give a definition.

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1 Notes on Gerhard Gentzen and the historical background

- mostly known for his "Hauptsatz"
- "Neben der Dozententtigkeit und der Forschung an der Widerspruchsfreiheit der Mathematik leitete Gentzen in Prag eine Gruppe von Oberschülerinnen, die Berechnungen durchführten, wohl fr ballistische Studien zur so genannten Vergeltungswaffe 2 fr Werner Osenberg. Zumindest war dies die offizielle Begründung."
- in his last days (he died in Soviet war prison), he developed several notions of integration of logical formulae
- because of investigations in several compensation cases, his notes are accessible only to a very small degree
- the definite book on Gentzen was written by a non-mathematician (at a time where access to his possesions was still possible), so the mathematical content delivered is very scarce
- the last passage in his notebook: "Die Ableitungen, die verdammten formalen Ableitungen! Gott steh mir bei!", this makes it probable that there even exists a syntactic integral which would reach far beyond the work in this paper

More resources can be found in [Pedophilia in Mathematics]. From Gerhard Gentzen's last days, we see once again how closely genius and madness are neighbouring.

2 Definition of the Gentzen integrals

2.1 The definite Gentzen integral for classical first order logic

Our subject of inspection will rely on measure theory. In most cases, though, one need not use a computer algebra system or the like to compute it. This is true for the case that all variables have finite range (and the measure is not too exotic).

For each variable, we need a measure space with finite measure, which is assumed to be normed. Furthermore, all atomic predicates have to be measureable functions in the product space of all its argument variables.

As already stated, we handle the semantic of a formula and use a recursive approach. The key point is the following observation: **an evaluation of a characteristic function from the product space of all variables to $\{0, 1\}$ can be identified with the truth value of a formula under an assignment**. Thus, characteristic functions are manipulated up to the top level, where the measure of the characteristic function is computed.

Definition 1 *Let $n \in \mathbb{N}_0$ and $\phi_1(x_1, \ldots, x_n), \phi_2(x_1, \ldots, x_n)$ be first order logic formulae with a possible quantor-bound variable x_{n+1}. Let (X_i, σ_i, μ_i) be bounded measure spaces*

$\forall i = 1, \ldots, n+1$, where w.l.o.g. $\mu_i(X_i) = 1$ is assumed $\forall i = 1, \ldots, n+1$. Define $\mu := \mu_1 \times \cdots \times \mu_n$ as the product measure on $X_1, \ldots, X_n$, which is automatically normed. The definite Gentzen Integral $\int_\perp^\top$ is defined recursively as:

$$\int_\perp^\top \phi(x_1, \ldots, x_n) = \int_\mu \int_{\perp *}^{\top *} \phi(x_1, \ldots, x_n)$$

$$\int_{\perp *}^{\top *} \neg\phi(x_1, \ldots, x_n) = 1 - \int_{\perp *}^{\top *} \phi(x_1, \ldots, x_n)$$

$$\int_{\perp *}^{\top *} (\phi_1(x_1, \ldots, x_n) \wedge \phi_2(x_1, \ldots, x_n)) = \left(\int_{\perp *}^{\top *} \phi_1(x_1, \ldots, x_n)\right) \cdot \left(\int_{\perp *}^{\top *} \phi_2(x_1, \ldots, x_n)\right)$$

$$\int_{\perp *}^{\top *} (\phi_1(x_1, \ldots, x_n) \vee \phi_2(x_1, \ldots, x_n)) = \int_{\perp *}^{\top *} \neg(\neg\phi_1(x_1, \ldots, x_n) \wedge \neg\phi_2(x_1, \ldots, x_n))$$

$$\int_{\perp *}^{\top *} \forall_{x_{n+1} \in X_{n+1}} \phi_1(x_1, \ldots, x_{n+1}) = \left((x_1, \ldots, x_n) \mapsto \begin{cases} 1 & \text{if } \int_\perp^\top \phi(x_1, \ldots, x_{n+1}) = 1 \\ 0 & \text{otherwise} \end{cases}\right)$$

$$\int_{\perp *}^{\top *} \exists_{x_{n+1} \in X_{n+1}} \phi_1(x_1, \ldots, x_{n+1}) = \int_{\perp *}^{\top *} \neg\forall_{x_{n+1} \in X_{n+1}} \neg\phi_1(x_1, \ldots, x_{n+1})$$

The (temporarily used) integral over an atomic formula $\tau(x_1, \ldots, x_n)$ in the recursion is the characteristic function $X_1 \times \cdots \times X_N \to \{0, 1\}$ for the assignments that make it true.

Is this well defined? More specifically, do we always get a notion of "how true" a formula is? Yes we do, as this proposition proves.

Proposition 2 *This formal definition gives a function from the space of finite logical formulae to* $[0, 1]$.

Proof. An easy induction proof on the height of the syntax tree of the logical formula. Because we have restricted ourselves to finite formulae, the syntax tree always has finite depth (but a generalization on depth $< \omega_1$ would be interesting: see [Transfinite Evaluation]). Note that except in the base case and the quantifier case, no integration has to be carried out. □

Theorem 3 *If all sets of variables that appear in the formula are finite and all elements have rational measure, then the Gentzen integral can be computed by a Siemens punch card automaton(see [Kriegsgerät]).*

Proof. Inductive proof: assume we have been able to obtain a characteristic function f on $X_1 \times \cdots \times X_n$ and want to compute the integral over this. Furthermore, assume w.l.o.g. that the measures of the single elements are kept as numbers on seperate punch cards (that is, the numerators of the smallest common denominator). Add the measure of all points a for which $f(a) = 1$ onto a new punch card and return this punch card together with another one for the least common denominator; this pair represents the rational number that we looked for. The other cases are even more easy: the multiplication of two punch cards corresponds to the logical "and", the negation (defined by $1 - \ldots$) is implemented by inverting the punched holes onto a new card. The other cases can be reduced to these. □

With the ubiquity of computer algebra systems, the preceding theorem is, of course, more of historic relevance. As punch cards were used in the German military, this is the use that Gentzen probably envisioned. But it also shows that the Gentzen integral can be used as a toy in a wealthy kindergarten.

The constructed integral is a semantic one, as we already noticed at several earlier occasions. As a corollary, we get:

Theorem 4 *(Gentzen-Forstenholz) The truth value of a statement does not depend on its wording. More formally said, the definite Gentzen integral for classical first order logic is constant on equivalence classes of logically equivalent formulae.*

Proof. As we have defined it above, our Gentzen integral relies only on the semantic of a formula, that is on sets of assignments that fulfill it (up to null sets). Per definition, two formulae are logically equivalent if they have the same semantic. □

One must not neglect the importance of this result. It says no less than that with the Gentzen integral, we can obtain a notion of truth of a statement, no matter how elegantly or clumsily it has been stated. That means one does not have to fear people that can convince others by solely using advanced rhetoric: as long as the others accept the mathematical rigorosity behind the Gentzen integral, they have no option but to debunk the phoney liar. If word about this concept can be spread into the general public, there will be a completely different discussion culture: nobody would enter a political talkshow without his Gentzen integral pocket calculator and the only real arguments could be over how you define an atomic function or a statement.

That, of course, is still a delicate matter, but I stay convinced:

Conjecture 5 *The rise of the definite Gentzen integral will bring a new Golden Age to our world.*

2.2 The general Gentzen integral and the Hegelian case

Until now, we remained in the tight borders of aristotelian logic together with tertium non datur, that means $A \vee \neg A$ for all statements A (at all single assignments). But what about more generality? There are situations where we do not simply want characteristic functions to be an indicator if an atomic formula is **completely** true for a special assignment, but **to what quantity** it is true. For this purpose, a slight and canonic generalization of the definite Gentzen integral can be used. It can be applied to any valued ring, i.e. to any ring R with an absolute value function $val : R \to \mathbb{R}_{\geqslant 0}$.

Definition 6 *We have the same setting as in the definition of the definite Gentzen integral. Let (R, val) be a valued ring, i.e. a quasi-probability ring. Let $C \in \mathbb{N}_0$. The generalized definite Gentzen integral is defined in exaclty the same manner as the definite Gentzen integral, aside from the following differences:*

$$\int_{\bot}^{\top} \phi(x_1, \ldots, x_n) = \int_{\mu} val\left(\int_{\bot *}^{\top *} \phi(x_1, \ldots, x_n)\right)^2$$

$$\int_{\bot *}^{\top *} \neg\phi(x_1, \ldots, x_n) = 1 - \left(\int_{\bot *}^{\top *} \phi(x_1, \ldots, x_n)\right)^C$$

$$\int_{\bot *}^{\top *} \forall_{x_{n+1} \in X_{n+1}} \phi_1(x_1, \ldots, x_{n+1}) = \left((x_1, \ldots, x_n) \mapsto \begin{cases} \int_{\bot}^{\top} \phi(x_1, \ldots, x_{n+1}) & \text{if it is} \geqslant 1 \\ 0 & \text{otherwise} \end{cases}\right)$$

Atomic formulae are now measureable functions $X_1 \times \cdots \times X_n \to R$.

We have to pay attention: there is no guarantee that the integral is finite, as can easily be seen in the next example:

Example 7 *Take $(\mathbb{R}, \mathcal{B}, \mu)$ to be the well-known uniform distribution on $(0, 1)$. Let (R, val) be $(\mathbb{R}, |\cdot|)$ and $\phi(x) = \frac{1}{\sqrt{x}}$. Then*

$$\int_{\bot}^{\top} \phi(x) = \int_{\mu} \left| \int_{\bot *}^{\top *} \phi(x) \right|^2 = \int_{\mu} \left| \frac{1}{\sqrt{x}} \right|^2 = \int_{\mu} \frac{1}{x} = \infty$$

Because of this, we have to stick to something like our well-known "$L^2(\Omega)$"-space, but this time, for functions $\omega \to R$ instead of $\to \mathbb{R}$ or $\to \mathbb{C}$. Our valuation function makes this very easy, as we integrate over real numbers in any case. It follows:

Corollary 8 *The generalized definite Gentzen integral of an L^2-formula is a non-negative real number.*

Proof. $L^2(\Omega)$ is a vector space, so we can add and subtract as we want. Moreover, since we have H"older's inequality (easily proven) and measurability is preserved under multiplication, it follows that multiplication is also an allowed operation that does not lead out of $L^2(\Omega)$. □

What do we choose as the exponent C in the negation? Consider the Hegelian philosophy (generally, for German quotes in this text, the use of Google Translate is recommended):

> *Das +a und -a sind zuerst entgegengesetzte Gren berhaupt; a ist die beiden zum Grunde liegende ansichseiende Einheit, das gegen die Entgegensetzung selbst Gleichgltige, das hier ohne weiteren Begriff als tote Grundlage dient. Das -a ist zwar als das Negative, das +a als das Positive bezeichnet, aber das eine ist so gut ein Entgegengesetztes als das andere. - G.W.F. Hegel, [Hegel]*

covers this subject. as well as

> *Es ist eine der wichtigsten Erkenntnisse,... dass jede (der Reflexionsbestimmungen) in ihrem Begriffe selbst die andere enthlt, einzusehen und festzuhalten; ohne diese Erkenntnis lsst sich eigentlich kein Schritt in der Philosophie tun. - G.W.F. Hegel, [Hegel]*

"Negation of negation leads to higher insight" is one of the key elements of the Hegelian philosophy. Therefore, the parameter C defines the effectivity of negation. Because 2 is the only number to satisfy $\frac{1}{x} + \frac{1}{x} = 1$, this is the canonic choice for Hegelianism.

The two following results show the two groundbreaking lemmata on the path to identifying where exactly Hegelian statements live:

Lemma 9 *(Habermas, 1968) The ring of Hegelian statements has characteristic* $< 10^{10^{10^{10^{20}}}}$.

Proof. See [Habermas] □

Lemma 10 *(Habermas, 1970) The ring of Hegelian statements is a field containing* $\mathbb{R}$.

Proof. See [Habermas] □

Theorem 11 *(Spießburg-Raiffmeisen, 2003) The ring of Hegelian statements is* $\mathbb{C}$ *with the standard absolute value.*

Proof. See [Spießburg-Raiffmeisen] □

Although $C = 2$ can be used in other settings, this is by far the most important one. Note that the Hegelian definite Gentzen integral contains the standard one, if only atomic formulae from the universe to $\{0, 1\}$ are used.

Equipped with this knowledge, we can start to consider some examples.

3 Some examples

3.1 The pony race

Example 12

As Gerhard Gentzen was a devout fan of pony races, we want to consider the following statement: Let there be three ponies $p1, p2, p3$, with equal winning chances. If $p1$ wins, it will have pneumonia with probability $\frac{2}{3}$, if it loses it will have pneuomonia with probability $\frac{1}{5}$. What is the truth value of "after a race, p1 has pneumonia"?

The formalization of this statement is obviously $\phi(G, x1, x2) = (p1wins(G) \wedge PW(x1)) \vee (\neg p1wins(G) \wedge PNW(x2))$. G is the variable that gives the winner of the race, $x1$ and $x2$ are variables with the probability of getting sick in the corresponding cases. The measures and atomic formulae are constructed canonically (for example, x2 has exactly five possible values that are equally probable, and exactly one of them is mapped to 1 by PNW).

$$
\begin{aligned}
\int_{\perp}^{\top} \phi(G, x1, x2) &= \int_{\mu} \int_{\perp *}^{\top *} \Big((p1wins(G) \wedge PW(x1)) \vee (\neg p1wins(G) \wedge PNW(x2)) \Big) \\
&= 1 - \int_{\mu} \left(\left(1 - \int_{\perp *}^{\top *} (p1wins(G) \wedge PW(x1)) \right) \cdot \right. \\
&\qquad \left. \left(1 - \int_{\perp *}^{\top *} (\neg p1wins(G) \wedge PNW(x2)) \right) \right) \\
&= 1 - \int_{\mu} \left(\left(1 - \mathbb{1}_{(p1,x_{11}),(p1,x_{12})} \right) \cdot \left(1 - \mathbb{1}_{(p2,x_{21}),(p3,x_{21})} \right) \right) \\
&= 1 - \left(\int_{\mu} \left(1 - \mathbb{1}_{(p1,x_{11}),(p1,x_{12})} - \mathbb{1}_{(p2,x_{21}),(p3,x_{21})} + \right. \right. \\
&\qquad \left. \left. \mathbb{1}_{(p1,x_{11}),(p1,x_{12})} \cdot \mathbb{1}_{(p2,x_{21}),(p3,x_{21})} \right) \right) \\
&= \int_{\mu} \left(\mathbb{1}_{(p1,x_{11}),(p1,x_{12})} + \mathbb{1}_{(p2,x_{21}),(p3,x_{21})} \right) \\
&= \int_{\mu} \left(\mathbb{1}_{(p1,x_{11})} + \mathbb{1}_{(p1,x_{12})} + \mathbb{1}_{(p2,x_{21})} + \mathbb{1}_{(p3,x_{21})} \right) \\
&= \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{1}{5} = \frac{2}{9} + \frac{2}{15} = \frac{10+6}{45} = \frac{16}{45}
\end{aligned}
$$

This is exactly the same as we would have obtained by calculating with conditional probabilities. However, it has to be said that this is not the main usage for the definite Gentzen integral.

3.2 Two arithmetic formulae

Example 13

Let $X_1 = X_2 = [0,1]$ with the uniform distributions. How true is $x + y < 1$?

$$\begin{aligned}\int_{\perp}^{\top} (x+y<1) &= \int_{\mu}\int_{\perp}^{\top} (x+y<1) = \int_{\mu} \mathbb{1}_{\{(x,y)\; x\in[0,1]\wedge y<1-x\}} \\ &= \int_{\mu_1}\int_{\mu_2} \mathbb{1}_{\{(x,y)\; x\in[0,1]\wedge y<1-x\}} = \int_{\mu_1} (1-x) \\ &= 1 - \frac{1}{2}x^2\Big|_{x=0}^{x=1} = \frac{1}{2}\end{aligned}$$

OK, that was kind of expected. But let's regard $\exists_{x_2} : x_1 = x_2$ (in this case, it holds that $\mu = \mu_1$):

$$\begin{aligned}\int_{\perp}^{\top} (\exists_{x_2} x_1 = x_2) &= 1 - \int_{\mu_1} \left(\int_{\perp *}^{\top *} \forall_{x_2} x_1 \neq x_2 \right) \\ &= 1 - \int_{\mu_1} \left(x_1 \mapsto \left(\begin{cases} 1 & \text{if } \int_{\perp}^{\top} x_1 \neq x_2 = 1 \\ 0 & \text{otherwise} \end{cases} \right) \right) \\ &= 1 - \int_{\mu_1} \left(x_1 \mapsto \left(\begin{cases} 1 & \text{if } \int_{\mu_1 \times \mu_2} \mathbb{1}_{\{(x_1,x_2);x_1\neq x_2\}} = 1 \\ 0 & \text{otherwise} \end{cases} \right) \right) \\ &= 1 - \int_{\mu_1} (x_1 \mapsto 1) = 0\end{aligned}$$

Why is this? Well, we have defined the Gentzen integral by measure theory. And for the construction of the quantifiers, it makes no difference whether the function evaluated in the intergral is nonzero at a null set.

3.3 To Be or Not to Be

One application for the generalized definite Gentzen Integral in the Hegelian philosophy is to look at a typically philosophical question. For example, what can we tell old Shakespeare in response to his century-old question? If we define (as is widely acknowledged in the mathematical philosophy community) the constant $Be : () \mapsto i$ (if no variable is used, then the canonic way is to define a variable with one possibility that has measure 1), then

$$\begin{aligned}\int_{\perp}^{\top} (Be \vee \neg Be) &= \int_{\mu} \left| 1 - \int_{\perp *}^{\top *} (\neg Be \wedge Be) \right|^2 = \int_{\mu} \left| 1 - (1 - i^2) \cdot i \right|^2 \\ &= \int_{\mu} \left| 1 - 2i \right|^2 = \int_{\mu} (1^2 + 2^2) = \int_{\mu} 5 = 5\end{aligned}$$

So, the Gentzen integral permits insight for which philosophers have strived for for centuries:

$$\int_{\perp}^{\top} (Be \vee \neg Be) = 5$$

Gentzen integral

During the computation, we have seen that "To Be and Not to Be" also has a nonzero probability. But that fits well into the Hegelian philosophy, where some statement and its negation can be true at the same time.

> Neu bei Hegel ist allerdings das dialektische: Subjektiver Geist und objektiver Geist sind identisch und gleichzeitig nicht identisch!
>
> Peter Möller, ber Hegel [Peter Möller]

In this light, Wette's theorem doesn't seem too unlikely to be true. Perhaps mathematics was, until now, indeed working in a "prison" of aristotelic logic. Knowing the results from an advanced seminar on our university that accomplished numerous breakthroughs in Wette theory (which was run by Maximum Likelihood, see [Wette-Seminar]), we tend to suggest that this might actually be the case.

Conjecture 14 *By the year 2050, everybody in mathematics has accepted Hegelian logic and the names of Aristoteles and Plato will only be known to extremely specialized researchers of the history of (failed) approaches to logic.*

4 Concluding remarks

4.1 Outlook

We have reconstructed an important tool in mathematical logic which will have implications on the society that can hardly be estimated at the moment: the definite Gentzen integral. We even have generalized it to include arbitrary valued rings (remind you, that includes quasi-probability rings!) as truth values instead of the classical, platonic $\{0, 1\}$. And we even have integrated the Hegelian logic (which seemed untameable even a decade ago) into our system. So what is left to be done?

Unfortunately, very much. Only finite formulae have been treated. Our construction was a purely semantic one which (at least in the non-generalized case) did not take the precise structure of the formula into account. Therefore, we cannot give something like explicit antiderivatives, which would be important for the recently flourishing field of differential logic. Especially the theory of logic differential forms in ω_0 variables cannot profit yet from the concept, as the semantic cannot be described in a finite form because of a theorem by Mulkowski.

In the theory of homological-topologic numeric transcendentally generated logical local algebrae, a tool that is similar to the Gentzen integral has been developed: the Hausdorff-Haustorf integral. The non-generalized definite Gentzen integral seems to be related in the following sense: it can be shown that the logical lifting of the logical projection of the logical lifting of the logical projection of the logical lifting of a homophoby is Gentzen-integrable exactly if its local field to the biggest prime number imaginable has κ-many elements, where κ is the smallest unreachable cardinal number.

4.2 Acknowledgements

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References

[Pedophilia in Mathematics] H. Gnabe-Rosemaritz: Pädophilie als Beweismethode in der Mathematik, *Kapitel 68: Gerhard Gentzen*

[Transfinite Evaluation] G. Meyer-Maier-Mayr: Recipes for evaluating transfinite algorithms

[Kriegsgerät] N. Spiegelwitz-Brauseknurr: Hochtechnologie des dritten Reiches und sein Nachwirken in Südamerika

[Hegel] G.W.F. Hegel: Werke in 20 Bänden. Band 6: Werke 6 = Wissenschaft der Logik, Band II

[Habermas] J. Habermas: Mathematisierung der Hegelei, III

[Spießburg-Raiffmeisen] A.H. Spieburg, E.G. Raiffmeisen: A theorem concerning complex philosophy. *Mathematische-Philosophische Annalen*, Bd. 13

[Peter Möller] P. Möller: http://philolex.de/hegel.htm

[Wette-Seminar] M. Likelihood: Transfinite Methode über endlichen Gruppen. http://www.cip.ifi.lmu.de/%7Esenjak/transgr/